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Large-time existence of the spherically symmetric flow of a self-gravitating viscous gas

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1 Introduction

We consider the spherically symmetric motion of a viscous and heat-conductive gas over a central rigid core (sphere). The gas is bounded by a free-surface, and let the boundaries be thermally-insulated. In the Eulerian coordinate such a motion is described by the following system of equations: for $r := |x|$, $x \in \mathbb{R}$

$$\begin{cases} \dfrac{D\rho}{Dt} = -\rho \dfrac{(r^2 v)_r}{r^2}, \\ \rho \dfrac{Dv}{Dt} = \left(-p + \zeta \dfrac{(r^2 v)_r}{r^2} \right)_r + \rho(f_c + f_g), \\ \rho \dfrac{De}{Dt} = \left(-p + \zeta \dfrac{(r^2 v)_r}{r^2} \right) \dfrac{(r^2 v)_r}{r^2} - 4\mu \left(\dfrac{(v^2)_r}{r} + \dfrac{v^2}{r^2} \right) - \dfrac{(r^2 q)_r}{r^2} \end{cases} \tag{1.1}$$

in $\mathfrak{D} := \cup_{t>0} (\mathfrak{D}_t \times \{t\})$, where $\mathfrak{D}_t := \{r \in \mathbb{R} \mid R_c < r < R(t)\}$ with the radius of the central core $R_c > 0$. The boundary conditions are

$$\begin{cases} \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = v(R(t), t) & \text{for } t > 0, \\ -p + \zeta \dfrac{(r^2 v)_r}{r^2} - 4\mu \dfrac{v}{r} = -p_e & \text{on } \varGamma_t, \ t > 0 \\ q = 0 & \text{on } \varGamma_t \cup \Sigma, \ t > 0 \\ v = 0 & \text{on } \Sigma, \ t > 0 \end{cases} \tag{1.2}$$

with $\varGamma_t := \{(r,t) \in \mathbb{R}^2 \mid r = R(t)\}$, $\Sigma := \{(r,t) \in \mathbb{R}^2 \mid r = R_c\}$. We seek to find, for any $t > 0$, the density $\rho = \rho(r,t)$, the velocity $v = v(r,t)$ (the velocity vector in $\mathbb{R}^3$ is $v\frac{x}{r}$) and the absolute temperature $\theta = \theta(r,t)$ together with the free-surface $R(t)$ satisfying (1.1) and (1.2) for given initial conditions

$$R(0) = R_0, \quad (\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0) = (\rho_0, v_0, \theta_0)(r), \quad r \in \overline{\mathfrak{D}_0}. \tag{1.3}$$

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Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r}$ is the material derivative; μ and ζ are coefficients of the shear and the bulk viscosity, respectively, satisfying $\mu > 0$ and $3\zeta - 4\mu \geq 0$; p_e is the external pressure assumed to be a non-negative constant. In this paper we consider only the case that the gas is ideal, *i.e.*, the pressure $p = p(\rho, \theta)$, and the internal energy per unit mass $e = e(\rho, \theta)$ are given by

$$p(\rho, \theta) = R\rho\theta, \quad e(\rho, \theta) = c_v\theta, \tag{1.4}$$

where R, c_v are the perfect gas constant, the specific heat capacity at constant volume, respectively, and that the heat-flux per unit area is given by the Newton-Fourier's law, $q\frac{x}{r} = -\kappa\nabla\theta$, *i.e.*, $q = q(r,t)$ has the form $q = -\kappa\theta_r$ with the heat-conductivity κ. For simplicity we assume that μ, ζ, c_v and κ are all positive constants. The self-gravitation of the gas per unit mass $f_g = f_g(r,t)$ and an attractive force due to the gravity of the core $f_c = f_c(r)$ are given by, respectively,

$$f_g(r,t) = -\frac{G_0}{r^2}\int_{R_c}^{r} 4\pi s^2\rho(s,t)\,\mathrm{d}s, \quad f_c(r) = -\frac{G_0M_0}{r^2} \tag{1.5}$$

with the Newtonian gravitational constant G_0 and mass of the core M_0.

In order to investigate our system of equations we introduce the Lagrangian-mass transformation. First, for each $(r,t) \in \cup_{t>0}(\overline{\mathfrak{D}^{(t)}} \times \{t\})$ let $\tilde{\xi}(r,t) := \lim_{\tau\to+0} X(\tau; r, t)$, where $X(\tau; r, t)$ is the solution of the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}X(\tau;r,t)}{\mathrm{d}\tau} = v(X(\tau;r,t),\tau), \quad \tau \in (0,t), \\ X(t;r,t) = r. \end{cases} \tag{1.6}$$

We see form (1.6) that a implicit function $r = \tilde{r}(\xi, t)$ determined by the equation $\xi = \tilde{\xi}(r,t)$ satisfies

$$\tilde{r}(\xi,t) = \xi + \int_0^t v(\tilde{r}(\xi,\tau),\tau)\,\mathrm{d}\tau. \tag{1.7}$$

Second, let $x = \tilde{x}(\xi) := \int_{R_c}^{\xi} \rho_0(s)s^2\,\mathrm{d}s$. Equation $x = \tilde{x}(\xi)$ can be solved inversely as $\xi = \eta(x)$ provided that ρ_0 is strictly positive function. From these change of variables the function $f = f(r,t)$ defined on $\overline{\mathfrak{D}}$ is transformed into $\check{f} = \check{f}(x,t) := f(\tilde{r}(\eta(x),t),t)$, and the function $f_0 = f_0(x)$ defined on $\overline{\mathfrak{D}_0}$ into $\check{f}_0 = \check{f}_0(x) := f_0(\eta(x))$. Putting $(v, v_0) = (v(x,t), v_0(x)) := (1/\check{\rho}(x,t), 1/\check{\rho}_0(x))$, $(u, u_0) = (u(x,t), u_0(x)) := (\check{v}(x,t), \check{v}_0(x))$, and omitting the every "$\check{\ }$", we deduce the following equations

$$\begin{cases} v_t = (r^2u)_x, \\ u_t = r^2\left(-R\dfrac{\theta}{v} + \zeta\dfrac{(r^2u)_x}{v}\right)_x - G\dfrac{M_c + x}{r^2}, \\ c_v\theta_t = \left(-R\dfrac{\theta}{v} + \zeta\dfrac{(r^2u)_x}{v}\right)(r^2u)_x - 4\mu(ru^2)_x + \left(\dfrac{r^4\kappa}{v}\theta_x\right)_x \end{cases} \tag{1.8}$$

in $\Omega \times (0,\infty)$ with $\Omega := (0, M)$, the boundary conditions

$$\begin{cases} \left(-R\frac{\theta}{v} + \zeta\frac{(r^2u)_x}{v} - 4\mu\frac{u}{r},\ \theta_x\right)\Big|_{x=M} = (-p_e, 0), \\ (u, \theta_x)\big|_{x=0} = (0,0), \end{cases} \tag{1.9}$$

and the initial conditions

$$(v, u, \theta)\big|_{t=0} = (v_0, u_0, \theta_0). \tag{1.10}$$

Here $M := \tilde{x}(R_0)$, $M_c := \frac{M_0}{4\pi}$, $G := 4\pi G_0$ and

$$r = r(x,t) = \left(R_c{}^3 + 3\int_0^x v(s,t)\,\mathrm{d}s\right)^{\frac{1}{3}},$$

which satisfies $r_x = \frac{v}{r^2}$ and $r_t = u$.

We impose on the initial data the compatibility conditions

$$\begin{cases} \left(-R\dfrac{\theta_0}{v_0} + \zeta\dfrac{(r_0{}^2u_0)'}{v_0} - 4\mu\dfrac{u_0}{r_0}\right)\Bigg|_{x=M} = -p_e, \\ u_0(0) = \theta_0{}'(0) = \theta_0{}'(M) = 0, \end{cases} \tag{1.11}$$

where $r_0 := \bar{r}[v_0]$ with, for an arbitrary function $V = V(x)$,

$$\bar{r}[V] := \left(R_c{}^3 + 3\int_0^x V(s)\,\mathrm{d}s\right)^{\frac{1}{3}}.$$

The temporally global solvability of the initial-boundary value problem (1.8)-(1.10) has already been investigated in [22]. Namely, by letting $Q_T := \Omega \times (0,T)$ one has

Theorem 1 (Umehara-Tani) *Let $\alpha \in (0,1)$. Assume that the initial data*

$$(v_0,\ u_0,\ \theta_0) \in C^{1+\alpha}(\overline{\Omega}) \times (C^{2+\alpha}(\overline{\Omega}))^2$$

satisfy (1.11) *and $v_0,\ \theta_0 > 0$ on $\overline{\Omega}$, and that $p_e > 0$, $3\zeta - 4\mu > 0$. Then there exists a unique solution (v, u, θ) of the initial-boundary value problem* (1.8)-(1.10) *such that*

$$(v,\ v_x,\ v_t,\ u,\ \theta) \in (C^{\alpha,\frac{\alpha}{2}}_{x,t}(\overline{Q_T}))^3 \times (C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}(\overline{Q_T}))^2$$

for any positive number T and the inequalities

$$\begin{cases} |v,\ v_x,\ v_t|^{(\alpha)}_{Q_T} + |u,\ \theta|^{(2+\alpha)}_{Q_T} \le C, \\ v,\ \theta \ge C^{-1} \ \ \text{in } \overline{Q_T} \end{cases} \tag{1.12}$$

hold, where C is a positive constant dependent on the initial data and T.

We note that in [22] radiating and reactive gases were also discussed. One easily sees that this theorem gives the solvability of the original problem formulated in the Eulerian coordinate. That is to say, one can construct the solution $R(t)$, (ρ, v, θ) of the initial-boundary value problem (1.1)-(1.3) with (1.4) and (1.5) such that, for $\mathfrak{D}_T := \bigcup_{t\in(0,T)} \left(\mathfrak{D}^{(t)} \times \{t\}\right)$,

$$R(t) \in C^{2+\frac{\alpha}{2}}([0,T]), \quad (\rho,\, \rho_r,\, \rho_t,\, v,\, \theta) \in (C_{y,t}^{\alpha,\frac{\alpha}{2}}(\overline{\mathfrak{D}_T}))^3 \times (C_{y,t}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\mathfrak{D}_T}))^2$$

for any positive number T by imposing suitable initial data corresponding to the one imposed in Theorem 1. For details, see the manner found in [23].

System of equations related to (1.8), (1.9) have been studied as some stellar models appearing in the astrophysical arguments (see, for example, [1,2,9]). One may say that mathematical studies concerning problems of compressible viscous fluid (especially, ideal gas case) in the spherically symmetric or one-dimensional framework have been based on the spirit of the works [8] and [7] due to Kazhikhov and Shelukhin, and Kazhikhov, respectively. Next in [11] Nagasawa developed the work [8] to the one applicable to the free-boundary case. Since in the astrophysical context it may not be reasonable to take some restrictions on the size of initial data, the works under large initial data mentioned above are also fundamental for our problem. Studies of temporally global behaviour of the viscous gases related to our problem are found, for example, in [24] discussed in a fixed annulus domain; in [12] in an unbounded domain around the core; in [5,6] including cases of both radiating and reacting gases; in [10,13,14] with contacting outer vacuume (in the sense of $p_e = 0$ in $(1.9)^1$). In the paper [3] the model closely-similar to ours was discussed. However, some statements and proofs in [3] seem not to be clear for the present author; for example, it looks rather hard to make hold the following estimate (Lemma 9 in [3]):

$$\int_0^t \int_0^M \theta {v_x}^2 \, \mathrm{d}x \, \mathrm{d}\tau \le C \quad \text{for any } t > 0 \tag{1.13}$$

with some constant $C > 0$ independent of time (which plays an important roll for the rest of estimates in that paper). The alternative to (1.13) is found in [21]. Although, recently the system of equations (1.8), (1.9) with $\mu = 0$, $\zeta = \zeta(v)$, $\kappa = \kappa(v, \theta)$ was investigated by Ducomet and Nečasová in [4], the asymptotic behaviour of the solution was not discussed in [4]. We also note that although the situations discussed in [5,6] seem rather general, the pure free-boundary case, *i.e.*, imposing Neumann boundary condition for the temperature on both two edges of the medium was not included.

2 Statements of theorems

Theorem 1 guarantees that the classical solution exists uniquely up to an arbitrary (large) fixed time T. However, since the estimates of the solution (1.12) is dependent on time T, those are not enough to investigate large-time behaviour of the solution. On the other hand, from physical point of view it is expected that the solution of our problem (1.8)-(1.10) converges, in some sense, to the corresponding stationary solution as time tends to infinity. In the paper [25], for a spherically symmetric flow of a barotropic viscous fluid the stationary problem was discussed, and then solved uniquely under a certain restricted condition on several physical parameters. In the present paper we obtain the condition similar to the one in [25] guaranteeing unique solvability of our stationary problem (see Proposition 1).

Before stating our result specifically, we briefly mention a theorem, obtained in [23], on the global behaviour of a one-dimensional gas flow having similar situation with ours. In this case, the equation of motion is given by in $\Omega_1 \times (0,\infty)$ with $\Omega_1 := (0,1)$,

$$u_t = \left(-R\frac{\theta}{v} + \mu\frac{u_x}{v}\right)_x - G\left(x - \frac{\int_0^M sv(s,t)\,\mathrm{d}s}{\int_0^M v(s,t)\,\mathrm{d}s}\right)$$

in the Lagrangian-mass coordinate. Let

$$g(a) := \left[\frac{5}{12} - a^2 + \frac{(a^2-\frac{1}{4})^2}{a}\log\left(1+\frac{1}{a-\frac{1}{2}}\right)\right]\frac{1}{a}\log\left(1+\frac{1}{a-\frac{1}{2}}\right),$$

$$h(b) := \sqrt{2b+\frac{1}{4}}.$$

The following condition confirms that the classical solution does not blow up at least under a certain condition. Namely, if the condition

$$(\tfrac{R}{c_v})^2(g\circ h)(\tfrac{p_e}{G}) < 1 \tag{2.1}$$

is satisfied, then there exists a positive constant C independent of T such that the inequality (1.12) holds for the solution of the one-dimensional problem. The functions $g(a)$ and $h(b)$ are derived from a calculation of the integrals containing $f(x) := p_e + \frac{G}{2}x(1-x)$:

$$\int_0^1 \frac{x^2(1-x^2)}{f(x)}\,\mathrm{d}x = \frac{2}{G}\left[\frac{5}{12} - a^2 + \frac{(a^2-\frac{1}{4})^2}{a}\log\left(1+\frac{1}{a-\frac{1}{2}}\right)\right],$$

$$\int_0^1 \frac{1}{f(x)}\,\mathrm{d}x = \frac{2}{G}\left[\frac{1}{a}\log\left(1+\frac{1}{a-\frac{1}{2}}\right)\right] \quad \text{with} \quad a = \sqrt{2\frac{p_e}{G}+\frac{1}{4}}.$$

For details, see [23]. We also note that the function $g(a)$ is monotone decreasing and convex downward with respect to $a \in (\frac{1}{2}, +\infty)$ with $\lim_{a\to\frac{1}{2}+0} g(a) = +\infty$ and $\lim_{a\to+\infty} g(a) = 0$. In the case that the condition (2.1) is satisfied, we proceed to further investigation concerning the temporally asymptotic behaviour of the flow. In fact, the solution $(v, u - \int_0^1 u\,\mathrm{d}x, \theta)$ of the one-dimensional problem converges to a steady state $(\tilde{v}, 0, \bar{\theta})$ in the sense of $H^1(\Omega_1)\cap C(\overline{\Omega}_1)$ as $t\to+\infty$. Moreover, there exist constants C and c independent of t such that for any $t>0$ the inequality

$$\left\|\left(v-\tilde{v},\, u - \int_0^1 u\,\mathrm{d}x,\, \theta-\bar{\theta}\right)(t)\right\| \le C\mathrm{e}^{-ct}$$

holds. Here

$$\tilde{v} = \tilde{v}(x) := \frac{R\bar{\theta}}{f(x)}, \quad \bar{\theta} := \frac{E_0}{c_v + R},$$

$$E_0 := \int_0^1 \left[\frac{1}{2}\left(u_0 - \int_0^1 u_0 \, dx\right)^2 + c_v\theta_0(x) + f(x)v_0(x) \right] dx.$$

Let us get back to our problem describing motion of a three-dimensional spherically symmetric gas flow. Our stationary problem is: To find the solution $(V, U, \Theta) = (V, U, \Theta)(x)$ of the boundary value problem

$$\begin{cases} (\bar{r}[V]^2 U)' = 0 & \text{in } \Omega, \\ \left(R\frac{\Theta}{V}\right)' = -G\frac{M_c + x}{\bar{r}[V]^4} & \text{in } \Omega, \\ -4\mu(\bar{r}[V]U^2)' + \left(\frac{\bar{r}[V]^4\kappa}{V}\Theta'\right)' = 0 & \text{in } \Omega, \\ U(0) = \Theta'(0) = \Theta'(M) = 0, \quad \left(R\frac{\Theta}{V} + 4\mu\frac{U}{\bar{r}[V]}\right)(M) = p_e. \end{cases}$$

We easily deduce that, if the solution exists, its form is limited to $(V(x;\bar{\Theta}), 0, \bar{\Theta})$ with an any positive constant $\bar{\Theta}$. Here $V(x;\bar{\Theta})$ means the solution V, for each (given) parameter $\bar{\Theta}$, of the following boundary value problem

$$\left(\frac{R\bar{\Theta}}{V}\right)' = -G\frac{M_c + x}{\bar{r}[V]^4} \quad \text{in } \Omega, \qquad \left.\frac{R\bar{\Theta}}{V}\right|_{x=M} = p_e. \tag{2.2}$$

Concerning the existence and uniqueness of the problem (2.2), we obtain the following result.

Proposition 1 *Assume that the condition*

$$p_e - \frac{GM(M_c + M/2)}{R_c{}^7} \cdot \frac{4RM\bar{\Theta}}{p_e} > 0 \tag{2.3}$$

is satisfied. Then there exists a unique solution $V \in C^1(\overline{\Omega})$ of the boundary value problem (2.2).

For proof of Proposition 1, see [20].

Here let us consider a function $\tilde{v} = \tilde{v}(x)$ and a positive constant $\bar{\theta}$ satisfying

$$\begin{cases} \dfrac{R\bar{\theta}}{\tilde{v}} = p_e + \displaystyle\int_x^M G\frac{M_c + s}{\bar{r}[\tilde{v}]^4} \, ds \qquad \text{for } x \in \overline{\Omega}, \\ \displaystyle\int_0^M \left(c_v\bar{\theta} + p_e\tilde{v} - G\frac{M_c + x}{\bar{r}[\tilde{v}]}\right) dx = E_0 \end{cases} \tag{2.4}$$

with

$$E_0 := \int_0^M \left(\frac{1}{2}{u_0}^2 + c_v\theta_0 + p_e v_0 - G\frac{M_c + x}{\bar{r}[v_0]} \right) \mathrm{d}x.$$

Our main result is

Theorem 2 *Let T be an arbitrary positive number, and α, μ, ζ and the initial data satisfy the hypotheses of Theorem 1. Assume that there exist a function $\tilde{v} \in C^1(\overline{\Omega})$ and a positive constant $\bar{\theta}$ satisfying the equalities (2.4), and that the condition*

$$p_e - \frac{R}{c_v} \cdot \frac{GM(M_c + M/2)}{{R_c}^4} \left(1 + \frac{4C_0}{{R_c}^3} \right) > 0 \tag{2.5}$$

is satisfied with

$$C_0 := \frac{1}{p_e} \left[E_0 + \frac{GM(M_c + M/2)}{R_c} \right].$$

Then there exists a positive constant C independent of T such that the inequality (1.12) holds for the solution (v, u, θ) of the initial-boundary value problem (1.8)-(1.10). Moreover, the solution (v, u, θ) converges to the state $(\tilde{v}, 0, \bar{\theta})$ as $t \to +\infty$ in the sense of $H^1(\Omega) \cap C(\overline{\Omega})$.

Remark 2.1 It is easily seen that if one can find $\tilde{v}$ (together with $\bar{\theta}$) satisfying (2.4), it also be a special solution $V(x; \bar{\theta})$ of the stationary problem (2.2). On the other hand, we see that the stationary solution of (2.2), whose unique existence is guranteed by Proposition 1, satisfies

$$MR\bar{\Theta} = \int_0^M \left(p_e V + G\frac{M_c + x}{\bar{r}^4} \int_0^x V \,\mathrm{d}s \right) \mathrm{d}x \tag{2.6}$$

with $\bar{r}$ which denotes $\bar{r}[V]$. Here let us assume that the solution $V(x; \bar{\Theta})$ also satisfies

$$Mc_v\bar{\Theta} + \int_0^M \left(p_e V - G\frac{M_c + x}{\bar{r}} \right) \mathrm{d}x = K_0 \tag{2.7}$$

for an any constant K_0. Combining (2.6) with (2.7) and putting $\lambda := \frac{1}{R_c}$, we consider the equation $g(\bar{\Theta}, \lambda) = 0$ on $\{(\bar{\Theta}, \lambda) \in \mathbb{R}^2 \,|\, \Theta > 0,\ \lambda \geq 0\}$, where

$$\begin{aligned} g(\bar{\Theta}, \lambda) &:= M(c_v + R)\bar{\Theta} \\ &\quad - \left\{ K_0 + \int_0^M \left(\lambda G \frac{M_c + x}{\hat{r}} + \lambda^4 G \frac{M_c + x}{\hat{r}^4} \int_0^x V \,\mathrm{d}s \right) \mathrm{d}x \right\} \end{aligned}$$

with $\hat{r} := \left(1 + 3\lambda^3 \int_0^x V(s; \bar{\Theta}) \,\mathrm{d}s\right)^{1/3}$. By noting that for $\bar{\Theta}_0 := \frac{1}{M}\frac{K_0}{c_v + R}$

$$g(\bar{\Theta}_0, 0) = 0, \quad \frac{\partial g}{\partial \bar{\Theta}}(\bar{\Theta}_0, 0) = M(c_v + R) > 0,$$

there exists a constant $\bar{\Theta} > 0$ near $\bar{\Theta}_0$ for suitably small $\lambda > 0$. Namely, we can say that in the situation both that (2.3) is satisfied and that R_c is sufficiently large, $\tilde{v}$ and $\bar{\theta}$ satisfying (2.4) are both determined.

Remark 2.2 Since an any constant $\bar{\Theta}$ satisfying (2.7) has the estimate

$$M\bar{\Theta} \le \frac{1}{c_V}\left[K_0 + \frac{GM(M_c + M/2)}{R_c}\right],$$

the inequality (2.3) is holded as far as we are in the situation of (2.5).

Proof of Theorem 1 was achieved by continuation of the temporally local solution with the help of suitable *a priori* estimates of the solution. We note that the fundamental theorem about local solvability of problems around ours had been already established, for example, in [15–17, 19], which are applicable to our problem without essential modifications. Concerning a priori estimates of the solution, we have (1.12).

To prove Theorem 2 we first establish the following proposition. For the norm

$$\|u\|_{E(Q_T)} := \|u\|_{E_1(Q_T)} + \|u_x\|_{E_1(Q_T)} + \|u_{xx}\|_{E_1(Q_T)} + \|u_t\|_{E_1(Q_T)},$$

$$\|u\|_{E_1(Q_T)} := \left(\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T \|u_x(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}$$

we obtain

Proposition 2 *For generalized derivatives*

$$(v_{xt},\ v_{tt},\ u_{xxx},\ u_{xt},\ \theta_{xxx},\ \theta_{xt}) \in (L^2(0,T;L^2(\Omega)))^6$$

of the solution (v,u,θ) *of the problem* (1.8)-(1.10), *there exists a constant* C *independent of* T *such that*

$$\begin{cases} \sup\limits_{t\in[0,T]} \|(v,\ v_x,\ v_t,\ v_{xt})(t)\| \le C, \\ \|u,\ \theta-\bar{\theta}\|_{E(Q_T)} + \displaystyle\int_0^T \|(v-\tilde{v},\ v_x-\tilde{v}_x,\ v_{tt})(t)\|^2 \,\mathrm{d}t \le C, \\ C^{-1} \le v,\ \theta \le C, \quad |u| \le C \quad \text{in } \overline{Q_T}. \end{cases}$$

Moreover,

$$\begin{aligned} &\sup_{x\in\overline{\Omega}} |u_x, \theta_x|(t) + \|(v-\tilde{v})(t)\|_{H^1(\Omega)\cap C(\overline{\Omega})} \\ &\qquad + \|(u,\ \theta-\bar{\theta})(t)\|_{H^2(\Omega)\cap C(\overline{\Omega})} + \|(u_t,\theta_t)(t)\| \longrightarrow 0 \quad \text{as } t\to+\infty. \end{aligned}$$

For proof of Proposition 2, see [21]. By virtue of Proposition 2 we obtain the Hölder estimates of the solution $(1.12)^1$ uniformly in time.

3 Notations

We introduce some function spaces. Let $\mathfrak{S} \subset \mathbb{R}$ be a domain, m a non-negative integer, $0 < \beta < 1$, and $0 < \sigma \leq 1$. We denote by $H^m(\mathfrak{S})$ the standard L^2-Sobolev space of order m equipped with the norm $\|\cdot\|_{H^m(\mathfrak{S})}$. For the case $m = 0$, we simply denote $\|\cdot\|$. For $u(x,t)$, an $H^m(\mathfrak{S})$-valued function of t, $\|u(t)\|_{H^m(\mathfrak{S})}$ means $\|u(\cdot,t)\|_{H^m(\mathfrak{S})}$. The space of continuous and bounded functions defined on $\overline{\mathfrak{S}}$ is denoted by $C(\overline{\mathfrak{S}})$. We use $C^{m+\beta}(\overline{\mathfrak{S}})$ as the space of functions $u = u(x)$ defined on $\overline{\mathfrak{S}}$ which have bounded derivatives up to order m on $\overline{\mathfrak{S}}$, and whose m-th order derivatives are uniformly Hölder continuous in $\overline{\mathfrak{S}}$ with exponent β. Its norm is defined by

$$|u|_{\mathfrak{S}}^{(m+\beta)} := \sum_{k=0}^{m} \sup_{x\in\overline{\mathfrak{S}}} |D^k u(x)| + \sup_{\substack{x', x''\in\overline{\mathfrak{S}} \\ x'\neq x''}} \frac{|D^m u(x'') - D^m u(x')|}{|x''-x'|^\beta} \quad \text{with } D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

Let T be a positive number. For a function $u = u(x,t)$ defined on $\overline{Q_T}$ we say that $u \in C_{x,t}^{\beta,\frac{\beta}{2}}(\overline{Q_T})$ if u is continuous in $\overline{Q_T}$ and has a finite norm

$$|u|_{Q_T}^{(\beta)} := |u|_{Q_T}^{(0)} + [u]_{x,Q_T}^{(\beta)} + [u]_{t,Q_T}^{(\frac{\beta}{2})},$$

where

$$\begin{aligned}
|u|_{Q_T}^{(0)} &:= \sup_{(x,t)\in\overline{Q_T}} |u(x,t)|, \\
[u]_{x,Q_T}^{(\sigma)} &:= \sup_{\substack{(x',t),\,(x'',t)\in\overline{Q_T} \\ x'\neq x''}} \frac{|u(x'',t) - u(x',t)|}{|x''-x'|^\sigma}, \\
[u]_{t,Q_T}^{(\sigma)} &:= \sup_{\substack{(x,t'),\,(x,t'')\in\overline{Q_T} \\ t'\neq t''}} \frac{|u(x,t'') - u(x,t')|}{|t''-t'|^\sigma}.
\end{aligned}$$

We also say that $u \in C_{x,t}^{2+\beta,1+\frac{\beta}{2}}(\overline{Q_T})$ if u is continuous in $\overline{Q_T}$ together with u_x, u_{xx}, u_t, and has a finite norm

$$|u|_{Q_T}^{(2+\beta)} := |u|_{Q_T}^{(0)} + |u_x|_{Q_T}^{(0)} + [u_x]_{t,Q_T}^{(\frac{1+\beta}{2})} + |u_{xx}|_{Q_T}^{(\beta)} + |u_t|_{Q_T}^{(\beta)}.$$

Remark 3.1 For the specific definitions of $C_{x,t}^{\beta,\frac{\beta}{2}}(\overline{\mathfrak{D}_T})$ and $C_{x,t}^{2+\beta,1+\frac{\beta}{2}}(\overline{\mathfrak{D}_T})$, the spaces of functions defined on a noncylindrical domain $\overline{\mathfrak{D}_T}$, see [18], Chapter VI.

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